Numerical methods

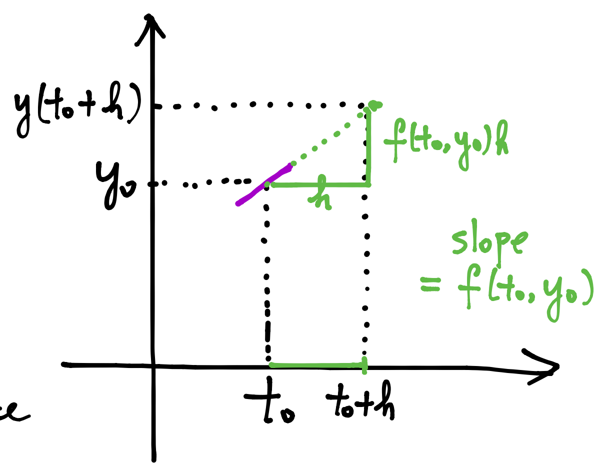
$y' = f(t,y)$

Applies to every typical first order ODE with initial values. Most useful when all analytic method fail. A priori, the IVP should be reasonably formulated

Euler's method

Idea: make use of direction field to approximate the solution by linear approximation

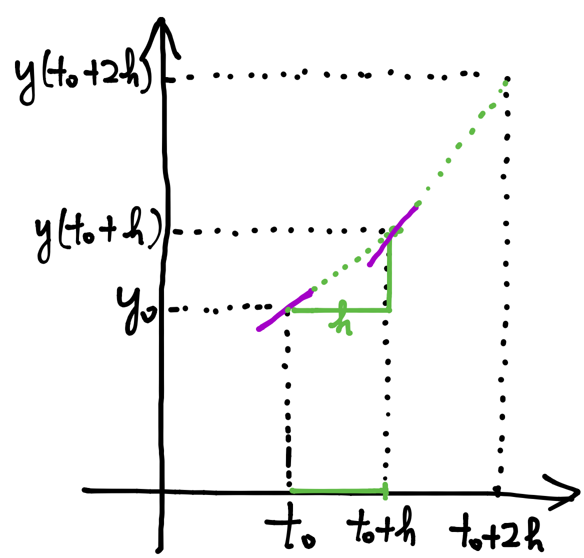
$$\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases}$$



Choose a small step-size h

Use linear approximation to estimate $y(t_0+h)$

$$y(t_0+h) = y_0 + h\cdot f(t_0, y_0)$$



Repeated apply linear approximation for sufficiently many times, we may get an estimate to any $y(t_0+nh)$ for $n = 1, 2, 3, \cdots$

Formula: Given the IVP, step-size h.

$$t_n = t_0 + nh, \quad n = 1, 2, 3, \cdots$$

Set y_n = approximation of $y(t_n)$, then y_n can be obtained from y_{n-1}, t_{n-1}, h by

$$y_n = y_{n-1} + h \cdot f(t_{n-1}, y_{n-1})$$

Example: $y' = t - y + 1$, $y(1) = 3$

Use Euler's method to estimate $y(3)$ with step-size $h = 0.5$

$h = 0.5$ $\quad y_n = y_{n-1} + h(t_{n-1} - y_{n-1} + 1)$

$t_{n-1} = t_0 + (n-1)h$ $\quad = y_{n-1} + 0.5\left(t_0 + (n-1)h - y_{n-1} + 1\right)$

$t_0 = 1$ $\quad = y_{n-1} + 0.5\left(1 + (n-1)0.5 - y_{n-1} + 1\right)$

$$= 0.5 y_{n-1} + 0.75 + 0.25n.$$

$t_0 = 1$, $t_1 = 1.5$, $t_2 = 2$, $t_3 = 2.5$, $t_4 = 3$ Need to find y_4

$y_0 = 3$

$y_1 = 0.5 y_0 + 0.75 + 0.25 \times 1 = 2.5$

$y_2 = 0.5 y_1 + 0.75 + 0.25 \times 2 = 1.25 + 0.75 + 0.5 = 2.5$

$y_3 = 0.5 y_2 + 0.75 + 0.25 \times 3 = 2.75$

$y_4 = 0.5 \times y_3 + 0.75 + 0.25 \times 4 = 1.375 + 1.75 = 3.125.$

General steps:

1. Formulate the formula and simplify y_n in terms of y_{n-1}, n, h.
2. Determine how many steps you want to compute
3. Compute using either calculator or math. software.

Attendance Quiz: Use Euler's method for the IVP

$$y' = t + y, \quad y(1) = 2$$

to estimate $y(3)$ with step size $h = 1$

$$\begin{aligned} y_{n+1} &= y_n + h f(t_n, y_n) \\ &= y_n + h\cdot(t_n + y_n) \\ &= y_n + h(t_0 + nh + y_n) \\ &= y_n + 1(1 + n + y_n) \\ &= 2y_n + n + 1 \end{aligned}$$

$f(t, y) = t + y$

$t_n = t_0 + nh$

$y(1) = 2 \Rightarrow t_0 = 1, \; y_0 = 2.$
$h = 1$

(replace n by $n-1 \Rightarrow y_n = 2y_{n-1} + n$)

Want $y(3)$, $t_0 = 1$, $t_1 = 2$, $t_2 = 3 \Rightarrow$ want y_2

n	t_n	y_n	y_{n+1}
0	1	2	$2\times 2 + 0 + 1 = 5$
1	2	5	$2\times 5 + 1 + 1 = 12$
2	3	12	$\Rightarrow y(3) \approx 12$

Error analysis:

For the example problem, $y(3) \approx 3.125$

The genuine solution is $y = t + 2e^{1-t}$

$$y(3) \doteq 3.27067$$

Approximation is not very good

However, if you use $h = \frac{1}{10}$. or $h = \frac{1}{10{,}000}$, approximation will be much better.

Recall: Taylor's theorem: $y(t_0+h) = y(t_0) + h\,y'(t_0) + \underbrace{\frac{h^2}{2} y''(t_0) + \cdots}_{\text{dropped}}$

Euler's method: $y(t_0+h) \approx y(t_0) + h \underbrace{f(t_0, y_0)}_{= y'(t_0)}$

The dropped term can be expressed using Peano remainder

Error $= \frac{1}{2} h^2 y''(\bar{t}_1)$. $\bar{t}_1$ some number between t_0, $t_0 + h$.

This tells

① If $y''(t)$ stays positive between t_0 & t_0+h, then the estimate will be smaller than the actual sol'n.

If y'' stays negative, estimate is larger.

(Not exclusive. Sometimes you can't tell)

② This error is the error in <u>one step</u>, called local truncation error. The error is proportional to h^2. In particular, lowering h means lowering the error.

③ The final error, after n step, will accumulate.

In general, if the local truncation error of some method is proportional to h^k, then the global truncation error will be proportional to h^{k-1}

In particular, GTE of Euler's method $\sim h$.

Example: $y' = t - y + 1$, $y(1) = 3$.

① Find out whether your estimate is larger or smaller than the actual solution.

② If the error of $y(3)$ is 0.2 with step-size $h = 0.5$

What would the error be if step-size $= 0.01$

$y' = t - y + 1$. To get y'', take derivative on both sides

$$y'' = 1 - y' \qquad \text{put in } y' = t - y + 1.$$

$$= 1 - (t - y + 1) = -t + y$$

within $[1, 3]$, we can estimate y'' using the value of each step

$y(1) = 3$, $y(1.5) \approx y_1 = 2.5$, $y(2) \approx y_2 = 2.5$, $y(2.5) \approx y_3 = 2.75$

$y(3) \approx y_4 = 3.125$

$y''(t_1) \approx -t_1 + y_1 = 2.5 - 1.5 = 1$, $y''(t_2) \approx 2.5 - 2 = 0.5$, $y''(t_3) \approx 2.75 - 2.5 = 0.25$

$y''(t_4) \approx y_4 - t_4 = 3.125 - 3 = 0.125$ ALL POSITIVE

So it's likely that the estimate is smaller.

WARNING: It is not very rigorous, just to give an idea how the error may be.

We know the global truncation error of Euler's method is proportional to h, i.e.

$$E = kh$$

Now $E = 0.2$ when $h = 0.5$, so $0.2 = k \cdot 0.5$

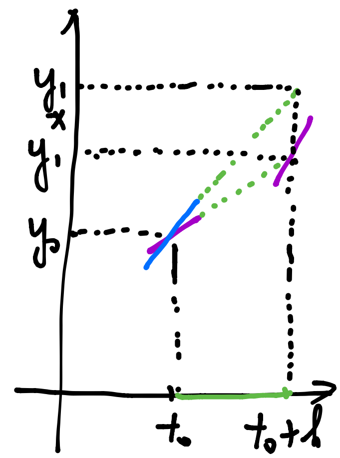
To find E when $h = 0.01$, we have $E = k \cdot 0.01$

Form the quotient:

$$\frac{0.2}{E} = \frac{k \cdot 0.5}{k \cdot 0.01} = 50 \Rightarrow E = 0.2 \times \frac{1}{50} = 0.004.$$

Improved Euler's method:

$$y' = f(t,y)\,,\quad y(t_0) = y_0$$



Formula: $t_n = t_0 + nh$

$$y_{n+1}^* = y_n + h\,f(t_n, y_n)$$

$$y_{n+1} = y_n + h\,\underbrace{\frac{f(t_n,y_n)+f(t_{n+1}, y_{n+1}^*)}{2}}_{\text{comes from modifying the slope}}$$

Error proportional to h^2 (More precise than Euler's method)

Remark: Although more precise, there are more to compute

Example: $y' = t - y + 1$, $y(1) = 3$ $h = 0.5$

$t_0 = 1$, $y_0 = 3$, $h = 0.5$

$$y_{n+1}^* = y_n + h f(t_n, y_n) = y_n + h\,(t_n - y_n + 1) = y_n + 0.5\,(1 + 0.5n - y_n + 1)$$

$$= 0.5\,y_n + 1 + 0.25n$$

$$y_{n+1} = y_n + \frac{h}{2}\left(f(t_n, y_n) + f(t_{n+1}, y_{n+1}^*)\right)$$

$$= y_n + \frac{h}{2}\left(t_n - y_n + 1 + t_{n+1} - y_{n+1}^* + 1\right)$$

$$= y_n + \frac{0.5}{2}\left(1 + 0.5n - y_n + 1 + 1 + 0.5(n+1) - y_{n+1}^* + 1\right)$$

$$= y_n + 0.25\left(4.5 + n - y_n - y^*_{n+1}\right)$$

$$= 0.75\,y_n - 0.25\,y^*_{n+1} + 1.125 + 0.25n$$

So the formulas to use are

$$y^*_{n+1} = 0.5y_n + 1 + 0.25n$$

$$y_{n+1} = 0.75y_n - 0.25y^*_{n+1} + 1.125 + 0.25n$$

Now compute

n	t_n	y_n	y^*_{n+1}	y_{n+1}
0	1	3	$0.5\times3+1+0.25\times0=2.5$	$0.75\times3-0.25\times2.5+1.125+0.25\times0$ $=2.75$
1	1.5	2.75	$0.5\times2.75+1+0.25\times1$ $=2.625$	$0.75\times2.75-0.25\times2.625+1.125+0.25\times1$ $=2.78125\doteq2.781$
2	2	2.781	$0.5\times2.781+1+0.25\times2$ $\doteq2.891$	$0.75\times2.781-0.25\times2.891+1.125+0.25\times2$ $\doteq2.988$
3	2.5	2.988	$0.5\times2.988+1+0.25\times3$ $\doteq3.244$	$0.75\times2.988-0.25\times3.244+1.125+0.25\times3$ $\doteq3.305$.

Compared to the actual solution 3.27067. it's more precise than the estimate given by Euler.

RK4-method (Runge-Kutta)

$$y' = f(t, y), \quad y(t_0) = y_0$$

Formula: $t_n = t_0 + nh$

$$k_1 = f(t_n, y_n)$$

$$k_2 = f(t_n + \tfrac{1}{2}h, \; y_n + \tfrac{1}{2}k_1 h)$$

$$k_3 = f(t_n + \tfrac{1}{2}h, \; y_n + \tfrac{1}{2}k_2 h)$$

$$k_4 = f(t_n + h, \; y_n + k_3 h)$$

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Elaborate analysis on Taylor's expansion.

Global truncation error proportional to h^4.

Maple lab 2 : numerical method

Lecture Notes of Differential Equation

Lecture	Page